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# On guarding the vertices of rectilinear domains

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**Abstract**

We prove that guarding the vertices of a rectilinear polygon $P$, whether by guards lying at vertices of $P$, or by guards lying on the boundary of $P$, or by guards lying anywhere in $P$, is NP-hard. For the first two proofs (i.e., vertex guards and boundary guards), we construct a reduction from minimum piercing of 2-intervals. The third proof is somewhat simpler; it is obtained by adapting a known reduction from minimum line cover.

We also consider the problem of guarding the vertices of a 1.5D rectilinear terrain. We establish an interesting connection between this problem and the problem of computing a minimum clique cover in chordal graphs. This connection yields a 2-approximation algorithm for the guarding problem.


*Keywords:* Geometric optimization; Guarding; NP-hardness; Approximation algorithms

## 1. Introduction

Problems dealing with visibility coverage are often called *art-gallery problems*. The "classical" art-gallery problem is to place guards in a polygonal region, such that every point in the region is visible to one (or more) of the guards. More formally, given a domain $P$, one needs to find a set $\mathcal{G}$ of points in $P$, of minimum cardinality, such that every point in $P$ is seen by at least one of the points, called *guards*, in $\mathcal{G}$. Often there are some restrictions on the location of the guards; e.g., guards may lie only at vertices (in which case they are called *vertex guards*).

The classical art-gallery problem, where guards may lie anywhere in the polygon or only at vertices, is known to be NP-hard, even if the underlying domain is a simple polygon [1,21,26]. Moreover, Eidenbenz et al. [11,12] have shown that these problems are APX-hard. Schuchardt and Hecker [28] proved that these problems remain NP-hard if we restrict our attention to (simple) rectilinear polygons. Their proof is based on a reduction from 3SAT.

In this paper we study two art-gallery problems. The first is the problem of guarding the *vertices* of a rectilinear polygon (GVRP) $P$. We consider three versions of this problem. In the first version guards may lie anywhere on

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the boundary of $P$ but not in the interior of $P$, in the second version guards may lie only at vertices of $P$, and in the third version guards may lie anywhere in $P$. We prove that despite the weaker requirement (i.e., only the vertices of $P$ must be guarded), the status of the problems does not change and all three versions remain NP-hard. For the first two proofs (i.e., boundary guards and vertex guards), we construct a reduction from minimum piercing of 2-intervals, where a 2-interval is the union of two disjoint line-segments on the real line. For the third proof, we construct a reduction from minimum line cover. (Note that minimum line cover has been used previously in hardness proofs for art-gallery problems by, e.g., Brodén et al. [4] and Joseph Mitchell. However, in order to use it in our setting, one needs sophisticated gadgets.)

The second problem that we study is that of guarding the vertices of a 1.5D rectilinear terrain. (A 1.5D rectilinear terrain is defined by an $x$-monotone chain $T$ of horizontal and vertical line segments; two vertices $u, v$ of $T$ see each other, if the line segment $\overline{uv}$ does not pass below $T$.) We establish an interesting connection between this problem and the problem of computing a minimum clique cover in chordal graphs (see below for the definition of chordal graph). This connection yields a 2-approximation algorithm for the guarding problem.

Ben-Moshe et al. [2] presented a constant-factor approximation algorithm for computing a set of guards for a 1.5D terrain that is defined by a *strictly* $x$-monotone polygonal chain. Their algorithm, however, cannot be applied (at least not immediately) to a 1.5D rectilinear terrain, since strict $x$-monotonicity is necessary at several places in their work. Moreover, the constant of approximation of their algorithm, as well as of the subsequent, purely theoretical, algorithm of Clarkson and Varadarajan [8], is big. Very recently King [19] gave a 4-approximation algorithm for minimum guarding of a 1.5D terrain. Again, strict $x$-monotonicity of the terrain is assumed. We also note that the idea of using perfect graph theory in the context of guarding is not new; see, e.g., [23].

*More related work*. Combinatorial art-gallery problems have been studied for three decades; see, e.g., [18,25,27,29] for surveys. The classical combinatorial result, the "art gallery theorem", states that $\lfloor n/3 \rfloor$ guards are sufficient and sometimes necessary to guard an $n$-vertex simple polygon [7]. Combinatorial results on the number of guards needed for various forms of guarding on terrains are given in [3].

Researches have mostly concentrated on obtaining good approximations. Ghosh [15] gave an $O(\log n)$-approximation for optimal guarding of a polygon by vertex guards, based on standard set cover results. Recent work [10,16] has focused on methods that efficiently apply the Brönnimann–Goodrich technique [5]. Efrat and Har-Peled [10] obtain an $O(\log k^*)$-approximation algorithm for simple polygon guarding with vertex guards, using time $O(n(k^*)^2 \log^4 n)$, where $k^*$ is the optimal number of vertex guards. Their technique can be applied to non-vertex guards, lying at points of a dense grid, adding a factor polylogarithmic in the grid density to the time bound. (No approximation algorithm is known if the guards are completely unrestricted and every point in the polygon must be guarded.) Their results apply also to polygons with holes and to 2.5D terrains, still with polylogarithmic approximation factors.

Very recently, Nilsson [24] presented a constant-factor approximation algorithm for guarding a monotone polygon. Using this algorithm, he also obtains an $O((c^*)^2)$-algorithm for guarding a rectilinear polygon, where $c^*$ is the size of an optimal guarding set.

Finally, for 1.5D terrains (i.e., for an $x$-monotone polygonal chain), Chen et al. [6] claim that by modifying the hardness proof of [26] one can show that the problem is NP-hard; details are omitted and are still to be verified.

## 2. GVRP is NP-hard

In this section we show that all three versions of GVRP are NP-hard. We begin with the version where guards may lie only on the boundary of the polygon.

### 2.1. *Guards may lie only on the boundary*

We show that if the guards are restricted to lie on the boundary of the polygon, then GVRP is NP-hard. We construct a reduction from minimum piercing of 2-intervals.

#### 2.1.1. *The 2-interval piercing problem*

A 2-interval $o$ is the union of two line-segments $t_a$ and $t_b$ on the $x$-axis, that can be separated by a vertical slab of constant width $c_0$. The minimum 2-interval piercing problem is defined as follows. Let $O$ be a set of $n$ 2-intervals. Find a set $\mathcal{P}$ of points on the $x$-axis, such that (i) for each 2-interval $o \in O$ there exists a point $p \in \mathcal{P}$ that pierces $o$

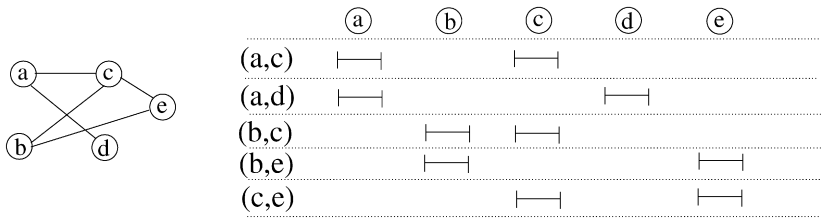


Fig. 1. Proof of Lemma 2.1: A reduction from vertex cover.

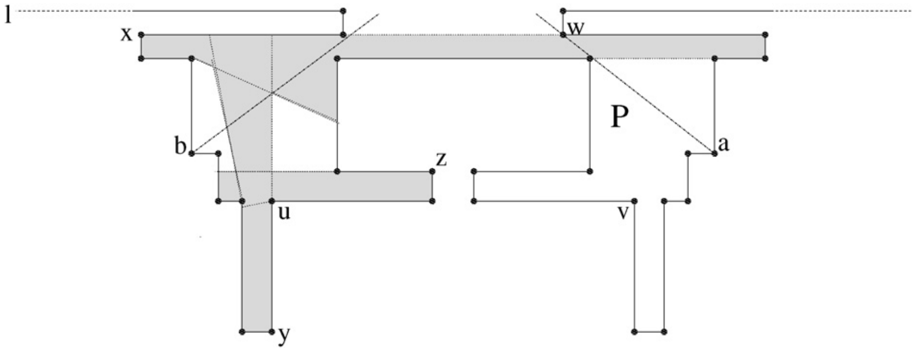


Fig. 2. A d-gadget.

(i.e., that lies in $o$), and (ii) $\mathcal{P}$ is as small as possible. Let D2IP denote the corresponding decision problem, that is, given $O$ and an integer $k > 0$, decide whether there exists a piercing set for $O$ of cardinality $k$. For completeness we show that D2IP is NP-hard, although we suspect that it is well known.

**Lemma 2.1.** *D2IP is NP-Hard.*

**Proof.** We construct a reduction from the decision version of vertex cover; see Fig. 1. Given a graph $G = (V, E)$, construct for each edge $e = (u, v) \in E$ a 2-interval $o_e$, such that its two line segments represent the two vertices $u$ and $v$, respectively. Let $O$ denote the set of 2-intervals that is obtained. It is clear that there exists a vertex cover of size $k$ if and only if there exists a piercing set for $O$ of size $k$. $\square$

*2.1.2. Reduction from D2IP*

We first present the gadget that we shall use. We call it *d-gadget* (short for double gadget), see Fig. 2. Any guard below the line $l$ is *local*. Some of the vertices of a d-gadget can only be guarded by a local guard (e.g., vertices $x$, $y$, and $z$). It is easy to see that in order to guard all these vertices one needs at least 3 local guards. However, any 3 local guards that guard all these vertices cannot see both $a$ and $b$. Moreover, one can locate 3 local guards on the boundary of a d-gadget, such that all the vertices of the d-gadget are guarded except for either $a$ or $b$. (E.g., locate 3 guards at the vertices $u$, $v$, and $w$, respectively.) Thus, another guard is required in order to guard all the vertices of a d-gadget. This guard does not have to be local; it can lie anywhere on the portion of the boundary of the polygon that is seen from the unguarded vertex $a$ or $b$.

We define a reduction function $f$ from D2IP to GVRP with boundary guards. Given an instance $\{O, k\}$ of D2IP, $f$ constructs a rectilinear polygon $P$, such that the vertices of $P$ can be guarded by $3|O| + k$ boundary guards if and only if there is a piercing set for $O$ of size $k$. In particular, $f$ constructs a rectilinear polygon $P$ with $|O|$ d-gadgets (see Fig. 3). The length of the top edge $t$ of $P$ is determined by the 2-intervals in $O$. For each 2-interval $o \in O$, $o = \{t_a, t_b\}$, $f$ constructs a d-gadget $g$ below the line $l$. $f$ locates $g$ and adjusts it, so that the vertex $a$ (resp. $b$) is (boundary) seen from outside of $g$ by any point on $t_a$ (resp. $t_b$) and only by these points. Fig. 4 shows such a construction. The portion of $t$ that is visible from vertex $a$ (resp. $b$) can be controlled by setting the auxiliary lines $a_l$ and $a_r$ (resp. $b_l$ and $b_r$). (Recall that the distance between $t_a$ and $t_b$ is at least $c_0$.)

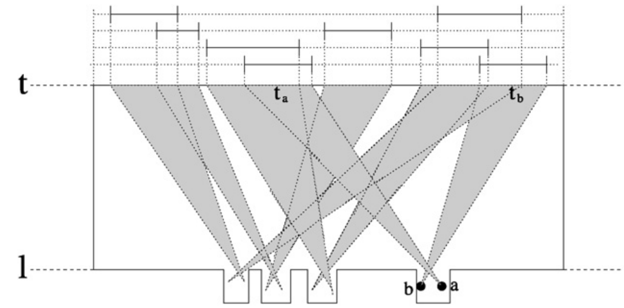


Fig. 3. The rectilinear polygon $P$. Each of the "holes" on the bottom represents a d-gadget.

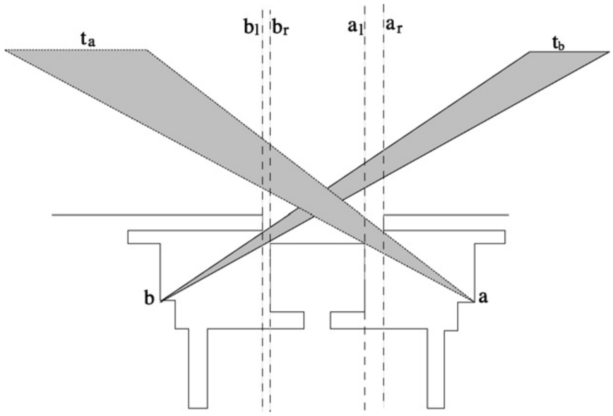


Fig. 4. The portion of $t$ visible from $a$ (resp. $b$) can be adjusted by setting the auxiliary lines $a_l$ and $a_r$ (resp. $b_l$ and $b_r$).

**Lemma 2.2.** *The rectilinear polygon $P$ that is obtained can be guarded by $3|O| + k$ boundary guards if and only if there exists a piercing set for $O$ of size $k$.*

**Proof.** Assume first that there exists a piercing set for $O$ of size $k$. We describe how to guard the vertices of $P$ with $3|O| + k$ boundary guards. For each point $p$ in the piercing set, we locate a guard at $p$ (which is of course on $t$). By the construction above, these $k$ guards see at least one of the vertices $g_a$,$g_b$ in each of the $|O|$ d-gadgets. In addition, these guards see all vertices of $P$ that are not below the line $l$. Finally, as explained above, one can locate, in each of the d-gadgets, 3 local (boundary) guards that see all the rest of the vertices of this d-gadget. Hence, the total number of guards is $3|O| + k$.

Assume now that the vertices of $P$ can be guarded by $3|O| + k$ boundary guards. We show a piercing set for $O$ of size $k$. As we argued above, each d-gadget requires at least 3 local guards. For each d-gadget $g$ that is guarded by more than 3 local guards, we replace these local guards by 3 local boundary guards at $u$, $v$, and $w$ (see Fig. 2) that see all the vertices of $g$ except for the vertex $g_a$, and by a guard in $t_a$. Hence, we have at most $k$ guards that are located on the top edge $t$. These guards constitute a piercing set for $O$, since, for each d-gadget $g$, at least one of the vertices $g_a$, $g_b$ is seen by a guard on $t$. In other words, for each 2-interval $o \in O$, there is a guard on $t$ that lies in $o$. □

The following theorem summarizes the result of this subsection.

**Theorem 2.3.** GVRP *with boundary guards is NP-hard.*

*2.2. Guards may lie only at vertices*

We show that if the guards are restricted to lie at vertices of the polygon, then GVRP remains NP-hard. As in the previous subsection (boundary guards), we construct a reduction from D2IP.

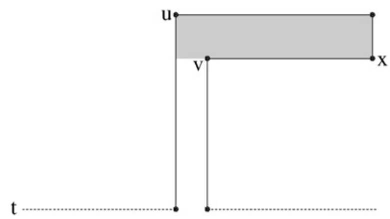


Fig. 5. An ear gadget.

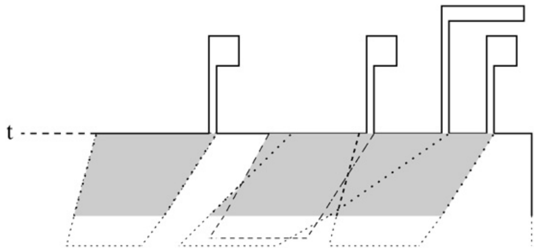


Fig. 6. An ear gadget is attached at each right endpoint of a segment on $t$.

### 2.2.1. Reduction form D2IP

In addition to d-gadgets, we shall also use *ear gadgets*, see Fig. 5. The vertices of an ear gadget can be guarded by a single guard that is located in the shaded rectangle (e.g., by a guard that lies at one of the vertices $u$ or $v$). Moreover, any set of guards that sees all the vertices of an ear gadget must include a guard in the shaded rectangle.

We define a reduction function $f$ from D2IP to GVRP with vertex guards. Given an instance $\{O, k\}$ of D2IP, $f$ constructs a rectilinear polygon $P$, such that the vertices of $P$ can be guarded by $m + 3|O| + k$ vertex guards if and only if there is a piercing set for $O$ of size $k$, where $m \leqslant 2|O|$ is the number of different right endpoints of the line segments corresponding to the 2-intervals in $O$.

$f$ constructs a rectilinear polygon with $|O|$ d-gadgets, as in Section 2.1.2. In addition (see Fig. 6), $f$ attaches an ear gadget at each right endpoint of a line segment on $t$ (i.e., at each right endpoint of a line segment of a 2-interval in $O$).

**Observation 2.4.** *Let $x$ be any point on $t$ that pierces a subset of the line segments (corresponding to a subset of $O$). Then, we may move $x$ to the first vertex to its right (which is a vertex of an ear gadget), without exiting any of the segments in the subset.*

**Lemma 2.5.** *The rectilinear polygon $P$ that is obtained can be guarded by $m + 3|O| + k$ vertex guards if and only if there exists a piercing set for $O$ of size $k$.*

**Proof.** Assume first that there exists a piercing set for $O$ of size $k$. One can locate $3|O| + k$ guards, as described in the proof of Lemma 2.2, such that these guards see all vertices of $P$ except for two or more vertices in each of the ear gadgets. The $3|O|$ local guards can be placed at vertices. Let $p$ be one of the $k$ guards. According to the observation above, we can move $p$ to the first vertex to its right without "losing" any of the vertices $g_a$, $g_b$ that it sees. Thus, by placing $m$ additional guards, one per ear gadget, we obtain a set of vertex guards that sees all the vertices of $P$.

Assume now that the vertices of $P$ can be guarded by $m + 3|O| + k$ vertex guards. We have at least one guard in each ear gadget that cannot see any vertex below the line $l$. Hence, as explained in Lemma 2.2, we have a piercing set for $O$ of size $k$. □

The following theorem summarizes the result of this subsection.

**Theorem 2.6.** GVRP *with vertex guards is NP-hard.*

### 2.3. Guards may lie anywhere

We show that if the guards may lie anywhere in the polygon, i.e., both in the interior and on the boundary, then GVRP is NP-hard. We construct a reduction from the minimum line cover problem (MLCP).

#### 2.3.1. The minimum line cover problem

The minimum line cover problem is defined as follows. Let $\mathcal{L} = \{l_1, \ldots, l_n\}$ be a set of $n$ lines in the plane. Find a set $\mathcal{P}$ of points, such that for each line $l \in \mathcal{L}$ there is a point in $\mathcal{P}$ that lies on $l$, and $\mathcal{P}$ is as small as possible. Let DLCP denote the corresponding decision problem, that is, given $\mathcal{L}$ and an integer $k > 0$, decide whether there exists a cover of size $k$. DLCP is known to be NP-hard [22]. Moreover, MLCP was shown to be APX-hard [4,20].

#### 2.3.2. Reduction from DLCP

We first present the gadget that we shall use. We call it *s-gadget* (short for single gadget), see Fig. 7. Some of the vertices of a s-gadget (e.g., vertices $x$ and $y$) can only be guarded by a local guard (i.e., by a guard below the line $l$ through the two top vertices in Fig. 7). It is easy to see that in order to guard all these vertices one needs at least one local guard, and any single local guard that sees all these vertices cannot see $a$.

We define a reduction function $f$ from DLCP to GVRP with guards anywhere. Given an instance $\{\mathcal{L}, k\}$ of DLCP, $f$ constructs a rectilinear polygon $P$, such that the vertices of $P$ can be guarded by $n + k$ guards if and only if there is a cover for $\mathcal{L}$ of size $k$. Let $R$ be a large enough rectangle, such that all the vertices of the arrangement of $\mathcal{L}$ lie in the interior of $R$. For each line $l \in \mathcal{L}$, $f$ constructs a s-gadget $g$ at one of the endpoints of the line segment $l \cap R$, in such a way that the vertex $a$ of $g$ can be guarded from outside $g$ only from points on $l \cap R$, see Fig. 8. Let $P$ be the rectilinear polygon that is obtained.

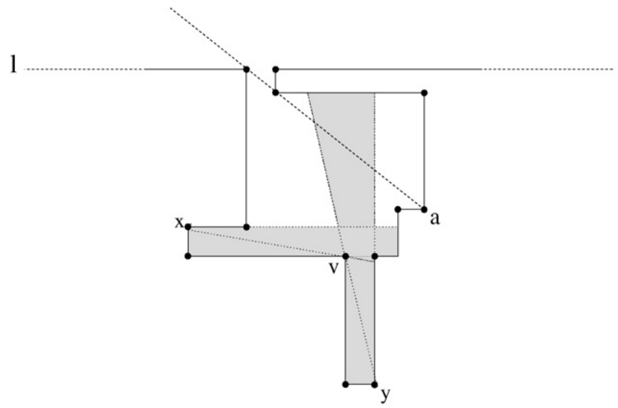


Fig. 7. A s-gadget.

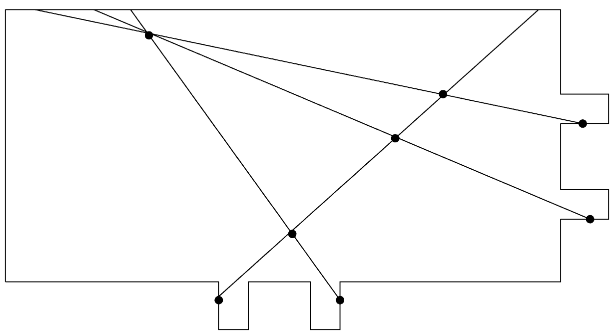

Fig. 8. The rectilinear polygon $P$. Each of the "holes" on the bottom and on the right represents a s-gadget.

**Lemma 2.7.** *The rectilinear polygon $P$ that is obtained can be guarded by $n + k$ guards if and only if there is a cover for $\mathcal{L}$ of size $k$.*

**Proof.** Follows immediately from the construction above. □

The following theorem summarizes the result of this subsection.

**Theorem 2.8.** GVRP *with guards anywhere in the polygon is NP-hard.*

## 3. Guarding the vertices of a 1.5D rectilinear terrain

A *1.5D terrain* (or simply, a *terrain*) $T$ is a polygonal chain specified by $n$ vertices $V(T) = \{v_1, \ldots, v_i = (x_i, y_i), \ldots, v_n\}$, such that $x_i \leqslant x_{i+1}$ (often strict monotonicity is assumed). The vertices induce $n - 1$ edges $E(T) = \{e_1, \ldots, e_i = (v_i, v_{i+1}), \ldots, e_{n-1}\}$. Let $p = (p_x, p_y)$ and $q = (q_x, q_y)$ be two points on $T$. We say that *$p$ sees $q$* (and *$q$ sees $p$*) if the line segment $\overline{pq}$ lies above $T$, or, more precisely, does not intersect the open region that is bounded from above by $T$ and from the left and right by the downwards vertical rays emanating from $v_1$ and $v_n$.

A terrain $T$ is a *1.5D rectilinear terrain* (or in short, a *r-terrain*) if each edge $e \in E(T)$ is either horizontal or vertical (and there are no two consecutive horizontal/vertical edges). A vertex $v_i$ of a r-terrain $T$ is *convex* (resp. *reflex*) if the angle formed by the edges $e_{i-1}$ and $e_i$ above $T$ is of 90 degrees (resp. 270 degrees). In r-terrains, we distinguish between two types of convex vertices—*left convex* and *right convex*. A convex vertex is left (resp. right) convex if $e_{i-1}$ (resp. $e_i$) is vertical. We denote the set of left convex vertices by $V_{lc}(T)$ and the set of right convex vertices by $V_{rc}(T)$. For example, in Fig. 9 vertex $a$ is reflex, vertex $b$ is left convex, and $c$ is right convex.

A set of points $\mathcal{G}$ on $T$ *guards* a set of points $V'$ on $T$ if each of the points in $V'$ is seen by at least one of the points (guards) in $\mathcal{G}$.

### 3.1. Some properties of terrains and r-terrains

In this section we explore some of the geometric properties of terrains and r-terrains. The following claim was stated and proved in [2].

**Claim 3.1.** *Let $a$, $b$, $c$ and $d$ be four points on a terrain $T$, such that $a_x < b_x < c_x < d_x$, where $q_x$ is the $x$-coordinate of point $q$. If $a$ sees $c$ and $b$ sees $d$, then $a$ sees $d$.*

One of the main differences between terrains and r-terrains is presented in the following trivial claim.

**Claim 3.2.** *Let $T$ be a r-terrain, $v \in V_{rc}(T)$, and $p$ a point on $T$. If $p$ sees $v$, then $p_x \leqslant v_x$.*

Clearly this is false for general terrains, where a convex vertex is not defined by a pair of orthogonal edges. Other unique properties of r-terrains are stated below.

**Claim 3.3.** *If a set $\mathcal{G}$ of points on a r-terrain $T$ guards a subset $V' \subseteq V_{lc}(T) \cup V_{rc}(T)$, then there exists a subset $\widehat{\mathcal{G}} \subseteq V(T)$ of reflex vertices, such that $\widehat{\mathcal{G}}$ guards $V'$ and $|\widehat{\mathcal{G}}| \leqslant |\mathcal{G}|$.*

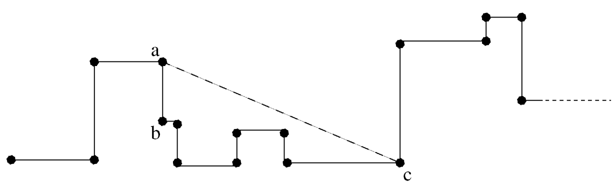


Fig. 9. A r-terrain; $a$ is reflex, $b$ is left convex, and $c$ is right convex; $a$ and $c$ see each other, but $b$ and $c$ do not.

**Proof.** A guard $g \in \mathcal{G}$ that lies in the interior of a horizontal edge can see only convex vertices with $y$-coordinate equal to that of $g$. Therefore, we may move each of these guards to either one of the endpoints of the edge on which it lies. Now, we move each guard in $\mathcal{G}$ that lies on a vertical edge (but not at the edge's reflex endpoint) to the edge's reflex endpoint. $\square$

**Claim 3.4.** *If $\mathcal{G} \subseteq V(T)$ guards all the convex vertices of a r-terrain $T$ (i.e., $\mathcal{G}$ guards the set $V_{lc}(T) \cup V_{rc}(T)$), then $\mathcal{G}$ guards all the vertices of $T$ (and all the vertical edges of $T$).*

**Proof.** Let $v \in V(T)$ be a reflex vertex. Then, at least one of its two neighboring vertices $u$ must be convex. It is easy to see that the guard in $\mathcal{G}$ that sees $u$ must also see $v$ (and the vertical edge $(u, v)$). $\square$

**Lemma 3.5.** *Let $u$, $v$ and $w$ be three right convex vertices of a r-terrain $T$, such that $u_x < v_x < w_x$. If there exist two vertices $g_1, g_2 \in V(T)$, such that $g_1$ sees both $u$ and $v$ and $g_2$ sees both $u$ and $w$, then there exists a vertex that sees all three vertices $u$, $v$ and $w$. Moreover, the one between $g_1$ and $g_2$ that precedes the other in the sequence of vertices defining $T$ is such a vertex.*

**Proof.** We first show that if $g_1$ lies to the left of $g_2$, then $g_1$ sees $w$. Consider the four vertices $g_1$, $g_2$, $v$ and $w$. We know that $g_{1x} < g_{2x} < v_x < w_x$. Since $g_1$ sees $v$ and $g_2$ sees $w$, we conclude by Claim 3.1 that $g_1$ also sees $w$. Assume now that $g_2$ lies to the left of $g_1$. If $g_1$ lies to the left of $u$, then we may conclude that $g_2$ also sees $v$, again by Claim 3.1. If, however, $g_1$ lies directly above $u$, then the vertices $u$, $g_1$, and $v$ are necessarily consecutive in the sequence of vertices defining $T$, and it is easy to see that in this case $g_2$ must also see $v$. Finally, if $g_{1x} = g_{2x}$, then the higher of the two (that is also the one that precedes the other in the sequence of vertices defining $T$) also sees the third vertex. $\square$

### 3.2. Guarding the vertices of a r-terrain

Let $T$ be a r-terrain. We present an algorithm that computes a set of (vertex) guards $\mathcal{G} \subseteq V(T)$ for $V(T)$ (i.e., each vertex in $V(T)$ is seen by a guard in $\mathcal{G}$), and prove that $\mathcal{G}$ is a 2-approximation, that is, $|\mathcal{G}| \leqslant 2m$, where $m$ is the size of an optimal set of guards for $V(T)$.

The algorithm computes optimal guard sets $\mathcal{G}_r$ for $V_{rc}(T)$ and $\mathcal{G}_l$ for $V_{lc}(T)$, and then outputs the set $\mathcal{G} = \mathcal{G}_r \cup \mathcal{G}_l$. According to Claim 3.4, $\mathcal{G}$ is a guard set for $V(T)$. Moreover, by Claim 3.3 $|\mathcal{G}_r|, |\mathcal{G}_l| \leqslant m$, and therefore $\mathcal{G}$ is a 2-approximation.

It remains to describe how to compute an optimal guard set for $V_{rc}(T)$ (alternatively, $V_{lc}(T)$). Although the final algorithm for computing such a guard set is simple and reminiscent of one of the base-case algorithms in [2], it is interesting, since it is the product of a connection that we discover between the problem of computing an optimal guard set for $V_{rc}(T)$ and the problem of computing a minimum clique cover for an appropriate chordal graph.

Several definitions are needed before we can proceed. A graph $G = (V, E)$ is *chordal* if every cycle of length four or more has a *chord*, that is, an edge that joins two non-consecutive vertices of the cycle. A *clique cover* of $G$ is a collection $V_1, \ldots, V_k$ of subsets of $V$, such that each of them induces a complete subgraph of $G$ (i.e., a clique) and $V_1 \cup \cdots \cup V_k = V$. In general, the minimum clique cover problem (i.e., compute a clique cover of minimum cardinality) is NP-hard [13]. However, if $G$ is chordal, then a minimum clique cover can be found in polynomial time [14].

We now construct a graph $G_r$ over the vertex set $V_{rc}(T)$. Draw an edge between two vertices $u, v \in V_{rc}(T)$ if and only if there exists a vertex $g \in V(T)$ that sees both $u$ and $v$. Next we claim that $G_r$ is chordal.

**Lemma 3.6.** *$G_r$ is chordal.*

**Proof.** Let $C = \{v_{i_1}, \ldots, v_{i_k}\}$ be a cycle of length at least four in $G_r$. Let $v$ be the leftmost vertex in $C$ and let $v', v'' \in C$ be its two adjacent vertices in the cycle. We know that there exists a vertex guard $g_1$ that sees both $v$ and $v'$, and a vertex guard $g_2$ that sees both $v$ and $v''$. Moreover, since $v$ is right convex (see Claim 3.2), $g_1$ and $g_2$ cannot lie to the right of $v$. Therefore, by Lemma 3.5, there exists a vertex guard $g$ that sees all three vertices $v$, $v'$, $v''$, implying that $C$ has a chord, namely, there exists an edge in $G_r$ between $v'$ and $v''$. $\square$

The following lemma, together with the fact that a minimum clique cover of $G_r$ can be computed in polynomial time, implies a polynomial time algorithm for computing an optimal guard set for $V_{rc}(T)$.

**Lemma 3.7.** *A subset $\widehat{V}$ of $V_{rc}(T)$ induces a clique of $G_r$ if and only if there exists a vertex guard in $V(T)$ that sees all the vertices in $\widehat{V}$.*

**Proof.** If $u \in V(T)$ sees all the vertices in a subset $\widehat{V}$ of $V_{rc}(T)$, then, by the definition of $G_r$, $\widehat{V}$ induces a clique of $G_r$. Assume now that $\widehat{V} \subseteq V_{rc}(T)$ induces a clique of $G_r$. If $|\widehat{V}| = 2$, then, by definition, there exists a vertex in $V(T)$ that sees both vertices in $\widehat{V}$. Assume therefore that $|\widehat{V}| \geqslant 3$. Let $u$ be the leftmost vertex, in the sequence of vertices defining $T$, that sees both the leftmost vertex $v_1$ in $\widehat{V}$ and another vertex $v_i$ in $\widehat{V}$. Let $v_j$ be any other vertex in $\widehat{V}$. Then since $\widehat{V}$ induces a clique, there must be a vertex $u' \in V(T)$ that sees both $v_1$ and $v_j$. According to Lemma 3.5 $u$ must also see $v_j$. □

*3.2.1. A direct algorithm for computing $\mathcal{G}_r$*

The algorithm for computing a minimum clique cover of a chordal graph [14], is based on the following two properties of chordal graphs. (i) Every chordal graph has a *simplicial vertex*; i.e., a vertex $v$ whose set of adjacent vertices forms a clique in the graph [9]. (ii) An induced subgraph of a chordal graph is chordal. Thus, given a chordal graph $G$, one can compute a minimum clique cover by repeating the following step until done: Find a simplicial vertex in the current subgraph (initially $G$), and remove it and its adjacent vertices from the subgraph.

Let $S$ be the set of simplicial vertices that were found during the execution of the algorithm. On the one hand, $S$ is an independent set of vertices, hence, a minimum clique cover of $G$ is of size at least $|S|$. On the other hand, each of the $|S|$ subsets that were removed during the execution of the algorithm forms a clique in $G$. Thus, these subsets constitute a minimum clique cover of $G$.

We now describe a direct algorithm for computing $\mathcal{G}_r$, an optimal guard set for $V_{rc}(T)$. Let $v$ be the leftmost vertex in $V_{rc}(T)$, and let $C_v \subset V_{rc}(T)$ be the subset of vertices $w$ for which there exists a guard in $V(T)$ that sees both $v$ and $w$. It follows (similar to the proof of Lemma 3.7) that there exists a single guard in $V(T)$ that sees all the vertices in $\{v\} \cup C_v$. We thus find such a guard $u$, and repeat the above step for the remaining unguarded vertices in $V_{rc}(T)$. Let $L$ be the set of left vertices that were found during the execution of the algorithm. Then, as in the algorithm for computing a minimum clique cover, at least $|L|$ guards are required to guard $V_{rc}(T)$, and since the algorithm finds exactly $|L|$ guards, it is optimal.

Finally, it is easy to implement the above guarding algorithm in $O(n^2)$ time, by first computing the (bipartite) visibility graph of the set of reflex vertices of $T$, on the one side, and the set $V_{rc}(T)$, on the other, using the algorithm of Hershberger [17]. The following theorem summarizes the result of this subsection.

**Theorem 3.8.** *Let $T$ be a 1.5D rectilinear terrain with $n$ vertices. One can compute in $O(n^2)$ time a set of guards $\mathcal{G}$ for $V(T)$ (and for all vertical edges of $T$), such that $|\mathcal{G}| \leqslant 2m$, where $m$ is the size of an optimal set of guards for $V(T)$.*

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